Exact Multiplets of Spontaneously Broken
Discrete Global Symmetries: the Example of $N = 2$ Susy QCD

ADEL BILAL

and

FRANK FERRARI

CNRS - Laboratoire de Physique Théorique de l'École Normale Supérieure[⋆]
24 rue Lhomond, 75231 Paris Cedex 05, France
bilal@physique.ens.fr, ferrari@physique.ens.fr

ABSTRACT

In these notes, we emphasize the rôle of spontaneous broken global discrete symmetries acting on the moduli space of $N = 2$ susy Yang-Mills theories and show how they can be used, together with the BPS condition, as a spectrum generating symmetry. In particular, in the strong-coupling region, all BPS states come in multiplets of this broken symmetry. This played a key rôle in the determination of the strong-coupling spectra.

To appear in the Proceedings of the Second International Sakharov Conference,
Moscow, May 1996, based on a talk given by A.B.

---

1. Introduction

It has long been known that the spontaneous breakdown of a continuous global symmetry in quantum field theory leads to certain predictions about the spectrum, namely the existence of massless Goldstone particles. Here, we will report on another phenomenon related to the spontaneous breakdown of a discrete global symmetry in theories having a manifold of physically inequivalent vacua.

It has been observed recently [1,2] that in $N = 2$ supersymmetric $SU(2)$ Yang-Mills theories with and without extra matter hypermultiplets, the BPS states in the strong-coupling region of moduli space come in *multiplets of such a broken discrete global symmetry.*

In the case of pure $N = 2$ susy $SU(2)$ Yang-Mills theory, for example, this discrete global symmetry is a $\mathbf{Z}_8$. A given vacuum is characterized by a non-vanishing expectation value $u = \langle \mathrm{tr}\, \phi^2 \rangle$, where ϕ is the scalar component of the $N = 2$ vector multiplet. Now only a $\mathbf{Z}_4$ subgroup leaves this u invariant while the remaining elements of $\mathbf{Z}_8$ act as a $\mathbf{Z}_2 : u \to -u$ and hence map a quantum field theory with a given vacuum (u) to another quantum field theory with a different vacuum $(-u)$. The important point is that the initial $\mathbf{Z}_8$ symmetry ensures that the theories at u and $-u$ are physically equivalent, and in particular have the same mass spectrum. This fact, together with the BPS mass condition, allowed us to show [1] that for every state at u of electric and magnetic quantum numbers n_e and n_m and mass m, there exists another state at $-u$ with different quantum numbers $\tilde{n}_e$, $\tilde{n}_m$ but same mass m. This then implies the existence of this state with quantum numbers $\tilde{n}_e$, $\tilde{n}_m$ at u, but with a different mass $\tilde{m}$. The two states (n_e, n_m) with mass m and $(\tilde{n}_e, \tilde{n}_m)$ with mass $\tilde{m}$ at the *same* point u of moduli space hence belong to a multiplet of the broken symmetry. In the weak-coupling region of moduli space, this chain of arguments continues and one predicts an infinite tower of states (unless $n_m = 0$). In the strong-coupling region, however, the multiplet is really just the above doublet, or actually a quartet if one distinguishes particles and antiparticles, forming a representation of the $\mathbf{Z}_8$. In the cases of $N_f = 1, 2, 3$ additional massless quark hypermultiplets, except for the replacement of this $\mathbf{Z}_8$ by $\mathbf{Z}_{4(4-N_f)}$, the structure is very similar [2]. In the remainder of this short note, we will try to give a flavour of how these properties come about, concentrating mainly on the pure $SU(2)$ susy Yang-Mills theory, and then outlining the generalizations to the cases with hypermultiplets.

2. BPS states and curve of marginal stability

For any background material on the $N = 2$ susy $SU(2)$ Yang-Mills theory we refer the reader to the original papers of Seiberg and Witten [3] or to [4]. Here, let us only mention that the $N = 2$ susy algebra admits two types of representations for massive states, namely long multiplets containing 16 helicity states, and short ones containing four. The masses of the short multiplets obey the so-called BPS condition $m = \sqrt{2}|Z|$ where Z is the central charge of the susy algebra. These states are called BPS states. It was shown in [3] that this central charge is given by

$$Z = n_e a(u) - n_m a_D(u) , \quad m = \sqrt{2}|Z| \tag{1}$$

where n_e and n_m are the integer electric and magnetic charge quantum numbers and $a(u)$ and $a_D(u)$ can be explicitly given in terms of hypergeometric functions (see e.g. [1,2,4]). We denote a BPS state by (n_e, n_m), so that e.g. the magnetic monopole is $(0, 1)$. The mass being given by the modulus of Z, the triangle inequality, together with the conservation of electric and magnetic charges, implies that a state with n_e and n_m relatively prime cannot decay into other states and hence is stable. This argument works as long as a_D is not a real multiple of a. If $a_D(u)/a(u)$ is real, decays are much easier to realise and otherwise stable BPS states can become unstable. The set $\mathcal{C}$ of all u on moduli space such that $a_D(u)/a(u) \in \mathbf{R}$ is called the curve of marginal stability [5,1]. It is almost an ellipse, symmetric about the origin $u = 0$, and goes through the singular points of moduli space $u = \pm 1$. As long as two points u and u' can be joined along a path not crossing $\mathcal{C}$, any (stable) BPS state (n_e, n_m) existing at u must also exist at u' and vice versa, since one can adiabatically deform the theory along that path on moduli space and the BPS state will always remain stable. This obviously is not true if the path has to cross the curve $\mathcal{C}$. Hence the curve separates the moduli space into two regions of constant BPS spectra. By the latter we mean the set of quantum numbers (n_e, n_m) that do exist, not the actual mass spectrum. We refer to the set of BPS states inside the curve as the strong-coupling spectrum and to the set outside the curve as the weak-coupling spectrum. We will show how these spectra are organized into multiplets of the broken discrete symmetry.

3. Broken discrete symmetries

The origin of the discrete symmetry is a continuous global $U(1)_R$ R-symmetry of the classical $N = 2$ susy Yang-Mills action, under which the scalar ϕ has charge two, transforming as $\phi \to e^{2i\alpha}\phi$. However, this symmetry does not survive quantization since the one-loop and instanton contributions are only invariant under a discrete subset with $\alpha = \frac{2\pi}{8}k$, $k \in \mathbf{Z}$. Thus only a $\mathbf{Z}_8$ subgroup remains from the original anomalous $U(1)_R$. This $\mathbf{Z}_8$ is a true quantum symmetry of the Hamiltonian and of the action. It acts as $\phi^2 \to (-)^k\phi^2$. A given vacuum has a non-vanishing expectation value of ϕ^2, i.e. $u = \langle \mathrm{tr}\,\phi^2 \rangle \neq 0$, and this spontaneously breaks $\mathbf{Z}_8$ to $\mathbf{Z}_4$, since it is clear that only those elements in $\mathbf{Z}_8$ corresponding to even k leave the vacuum invariant. The other elements, corresponding to the quotient $\mathbf{Z}_2$, act as $u \to -u$ and map a given vacuum to another, but physically equivalent vacuum.

This implies that for any BPS state (n_e, n_m) at u, there must be some (other) BPS state $(\tilde{n}_e, \tilde{n}_m)$ at $-u$ with the same mass:

$$|\tilde{n}_e a(-u) - \tilde{n}_m a_D(-u)| = |n_e a(u) - n_m a_D(u)| . \tag{2}$$

Since the mass formula is given in terms of the symplectic invariant of (n_e, n_m) and (a_D, a) it is clear that eq. (2) implies the existence of a matrix $G \in Sp(2, \mathbf{Z})$ such that

$$\begin{pmatrix} \tilde{n}_e \\ \tilde{n}_m \end{pmatrix} = \pm G \begin{pmatrix} n_e \\ n_m \end{pmatrix} \quad , \quad \begin{pmatrix} a_D \\ a \end{pmatrix}(-u) = e^{i\omega} G \begin{pmatrix} a_D \\ a \end{pmatrix}(u) \tag{3}$$

with $e^{i\omega}$ some phase. From the explicit expressions of a_D and a in terms of hypergeometric functions one finds[*] [1]

$$G = G_{W,\epsilon} \equiv \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \tag{4}$$

where $\epsilon = \pm 1$ depending on whether u is in the upper or lower half plane, and the subscript W refers to the weak-coupling region outside the curve $\mathcal{C}$, since inside the curve there is a slight subtlety discussed below. According to the argument just given, the existence of a BPS state (n_e, n_m) at u with mass m implies the existence of a BPS state $(\tilde{n}_e, \tilde{n}_m) = \pm G_{W,\epsilon}(n_e, n_m)$

---

[*] Note that the $\mathbf{Z}_2$ symmetry simply amounts to shifting the θ angle by 2π.

at $-u$ with the same mass m. But if u is outside the curve, then so is $-u$, and we know that the same state $(\tilde{n}_e, \tilde{n}_m)$ must also exist at u although with a different mass $\tilde{m}$. Thus the spontaneously broken symmetry $\mathbf{Z}_8$, through the broken generators leading to the $\mathbf{Z}_2$ acting on the moduli space, together with the BPS condition, allows us to deduce the existence of $G_{W,\epsilon}(n_e, n_m)$ from the existence of (n_e, n_m) at the *same* point of moduli space, i.e. for the same quantum field theory with the same vacuum. Given the form of the matrix $G_{W,\epsilon}$, starting from the monopole $(0, 1)$ this leads to a whole "infinite multiplet" containing all dyons $(n, 1)$. In this sense one might call the $\mathbf{Z}_2$ symmetry realised through the $G_{W,\epsilon}$ matrices a spectrum generating symmetry. On the other hand, the W-boson $(1, 0)$ is a singlet: $G_{W,\epsilon}(1, 0) = (1, 0)$.

4. Strong-coupling multiplets

Things become even more interesting in the strong-coupling region inside the curve $\mathcal{C}$, where other methods like semi-classical quantization do not apply. This region is separated into an upper and lower one due to the cut of the function $a(u)$ on the real line. As a consequence, one has to introduce two different descriptions of the same BPS state. If a state is described by (n_e, n_m) below the cut, it will be described by $M_1(n_e, n_m)$ above the cut, where $M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ is the monodromy matrix around $u = 1$. Repeating now the argument, the existence of (n_e, n_m) at u (with Im$u > 0$) implies the existence of $G_{W,\epsilon}(n_e, n_m)$ at $-u$. This same BPS state must then also exist at u, but at u it is described as $M_1 G_{W,\epsilon}(n_e, n_m)$. Similarly, if one starts with Im$u < 0$. Hence the weak-coupling G matrix gets replaced by

$$G_{S,\epsilon} = (M_1)^\epsilon G_{W,\epsilon} = \begin{pmatrix} 1 & \epsilon \\ -2\epsilon & -1 \end{pmatrix} . \tag{5}$$

This matrix has the property that $G_{S,\epsilon}^2 = -\mathbf{1}$. As a consequence, the strong-coupling multiplets are quartets (or doublets if one considers the hypermultiplets which contain both the particles and their antiparticles), and all strong-coupling hypermultiplets come in pairs. (Of course, there is also the everywhere massless "photon" vector multiplet with $n_e = n_m = 0$ which is a singlet.) For $\epsilon = +1$ e.g., these pairs are

$$\pm \begin{pmatrix} n_e \\ n_m \end{pmatrix} \quad \longleftrightarrow \quad \pm G_{S,+} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = \pm \begin{pmatrix} n_e + n_m \\ -2n_e - n_m \end{pmatrix} . \tag{6}$$

We insist that for a given point u these two states have different masses. The two states

4

responsible for the singularities on the moduli space, namely the magnetic monopole $\pm(0,1)$ and the dyon $\pm(1,-1)$, form such a $\mathbf{Z}_2$ doublet. In [1] it was shown that this is the only doublet in the strong-coupling spectrum, since for any other doublet (6) one or the other partner becomes massless somewhere on the curve $\mathcal{C}$ of marginal stability This would lead to extra singularities that we know are not there. Hence these other doublets cannot exist. This provided a simple determination of the strong-coupling spectrum [1]. Later on, these results were also obtained within a string theory context [6].

5. Inclusion of massless quark hypermultiplets

The previously discussed results have been generalized [2] to the inclusion of $N_f = 1, 2, 3$ massless quark hypermultiplets in the defining representation of the gauge group $SU(2)$ (asymptotically free theories). In each case, there is a similar curve of marginal stability separating the Coulomb branch of moduli space into a weak-coupling region (outside the curve) and a strong-coupling region (inside the curve). The $U(1)_R$ symmetry is anomalous, and so is the parity operation of the flavour symmetry $O(2N_f)$. The combined anomaly free subgroup is the discrete $\mathbf{Z}_{4(4-N_f)}$ acting again non trivially on ϕ, and the quantum flavour symmetry is $Spin(2N_f)$. The vacuum expectation value $u = \langle \operatorname{tr} \phi^2 \rangle$ spontaneously breaks $\mathbf{Z}_{4(4-N_f)}$ to $\mathbf{Z}_4$ with the quotient $\mathbf{Z}_{4-N_f}$ acting as a symmetry on the Coulomb branch of moduli space, relating physically equivalent theories. For $N_f = 1$ this is a $\mathbf{Z}_3$ acting as $u \to e^{2\pi i/3}u$, for $N_f = 2$ it is a $\mathbf{Z}_2$ acting as $u \to -u$, while there is no such symmetry for $N_f = 3$. As a consequence, the strong-coupling hypermultiplets form triplets for $N_f = 1$ and doublets for $N_f = 2$ [2]. In these two cases the strong-coupling spectrum just contains the one triplet or doublet that groups the states responsible for the singularities. For $N_f = 3$ again, the strong-coupling spectrum only contains the two states responsible for the singularities, but here they do not form any multiplet.

6. Conclusion

We hope to have convinced the reader of the extreme usefulness of the broken symmetries acting on moduli space, in particular in the strong-coupling regions where they allow for a simple determination of the spectra of BPS states.

REFERENCES

1. F. Ferrari and A. Bilal, *The strong-coupling spectrum of Seiberg-Witten theory*, to appear in Nucl. Phys. **B469** (1996), `hep-th/9602082`.

2. A. Bilal and F. Ferrari, *Curves of marginal stability, and weak and strong-coupling BPS spectra in $N = 2$ supersymmetric QCD*, École Normale Supérieure preprint LPTENS-96/22, `hep-th/9605101`.

3. N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. **B426** (1994) 19, `hep-th/9407087`, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys. **B431** (1994) 484, `hep-th/9408099`.

4. A. Bilal, *Duality in $N = 2$ susy $SU(2)$ Yang-Mills theory: A pedagogical introduction to the work of Seiberg and Witten*, École Normale Supérieure preprint LPTENS-95/53, `hep-th/9601007`.

5. P.C. Argyres, A.E. Faraggi and A.D. Shapere, *Curves of marginal stability in $N = 2$ super-QCD*, preprint IASSNS-HEP-94/103, UK-HEP/95-07, `hep-th/9505190`;
   A. Fayyazuddin, *Some comments on $N = 2$ supersymmetric Yang-Mills*, Mod. Phys. Lett. **A10** (1995) 2703, `hep-th/9504120`;
   M. Matone, *Koebe $1/4$-theorem and inequalities in $N = 2$ super-QCD*, Phys. Rev. **D53** (1996) 7354, `hep-th/9506181`.

6. A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, *Self-dual strings and $N = 2$ supersymmetric field theory*, preprint CERN-TH/96-95, HUTP-96/A014, USC-96/008, `hep-th/9604034`.